Review on Week 3

Limit Theorems

There are concepts and theorems concerning sequences and their limits. I shall list out some of which is important.

Definition (c.f. Definition 3.2.1)**.** A sequence (x_n) is said to be *bounded* if there exists a real number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Remark. It is equivalent to say that the set $\{x_n : n \in \mathbb{N}\}$ is bounded. (Try to prove it!)

Theorem (c.f. Theorem 3.2.2)**.** *A convergent sequence of real numbers is bounded.*

Example 1. Observe the following examples:

- The sequence $(1/n)$ is convergent, **hence** it is bounded by this theorem.
- The sequence $((-1)^n)$ is bounded but not convergent.
- The sequence (n) is not bounded, **hence** it is not convergent.

Can you find a sequence that is unbounded and convergent?

Theorem (c.f. Theorem 3.2.5)**.** *Let (x_n) and (y_n) be convergent sequences satisfying*

$$x_n \leq y_n, \quad \forall n \in \mathbb{N}.$$

Then $\lim(x_n) \leq \lim(y_n)$.

Theorem (c.f. Theorem 3.2.6)**.** *Let (x_n) be a convergent sequence satisfying*

$$a \leq x_n \leq b, \quad \forall n \in \mathbb{N}.$$

Then $a \leq \lim(x_n) \leq b$.

Squeeze Theorem (c.f. 3.2.7)**.** *Let (x_n), (y_n) and (z_n) be sequences. Suppose that*

$$x_n \leq y_n \leq z_n, \quad \forall n \in \mathbb{N}.$$

and $\lim(x_n) = \lim(z_n)$*. Then (y_n) is convergent and* $\lim(x_n) = \lim(y_n) = \lim(z_n)$.

Remark. Notice the following:

- The inequality sign "$\leq$" can be replaced by the strict inequality sign "$<$" in the **assumption** of each theorems. However, the signs **cannot** be replaced in the **conclusion**.
- Since only the tails of the sequences affect the limit behaviour, the condition in the assumption can be relaxed to hold for all $n \geq K$, where K is a natural number.

Example 2. Observe the following examples:

- Note that $1/n > 0$ for all $n \in \mathbb{N}$. Hence $\lim(1/n) \geq 0$. (But $\lim(1/n) > 0$ is false.)
- Note that $1/(n - 10.5) \geq 0$ for all $n \geq 11$, and $\lim(1/n) \geq 0$ still holds.
- $\lim(\sin n/n) = 0$ because

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

Limit Superior/Limit Inferior

Definition (c.f. Definition 3.3.1)**.** Let (x_n) be a sequence of real numbers.

- (x_n) is said to be *increasing* if $x_1 \le x_2 \le \cdots \le x_n \le x_{n+1} \le \cdots$.
- (x_n) is said to be *decreasing* if $x_1 \ge x_2 \ge \cdots \ge x_n \ge x_{n+1} \ge \cdots$.
- (x_n) is said to be *monotone* if it is either increasing or decreasing.

Monontone Convergence Theorem (c.f. 3.3.2)**.** *A monotone sequence (x_n) of real numbers is convergent if and only if it is bounded. Moreover,*

- *if (x_n) is increasing, then* $\lim(x_n) = \sup\{x_n : n \in \mathbb{N}\}$.
- *if (x_n) is decreasing, then* $\lim(x_n) = \inf\{x_n : n \in \mathbb{N}\}$.

Having the idea that bounded monotone sequences must have limit, can we construct monotone sequences from a given sequence?

Definition (c.f. Definition 3.4.10 and Theorem 3.4.11)**.** Let (x_n) be a **bounded** sequence of real numbers.

- The *limit superior* of (x_n), denoted by $\limsup(x_n)$ or $\overline{\lim}(x_n)$, is given by

$$\overline{\lim}(x_n) = \inf_n \left(\sup_{k \ge n} x_k \right) = \lim_{n\to\infty} \left(\sup_{k \ge n} x_k \right).$$

- The *limit inferior* of (x_n), denoted by $\liminf(x_n)$ or $\underline{\lim}(x_n)$, is given by

$$\underline{\lim}(x_n) = \sup_n \left(\inf_{k \ge n} x_k \right) = \lim_{n\to\infty} \left(\inf_{k \ge n} x_k \right).$$

Remark. The definitions of limit superior and limit inferior in the textbook are complicated and hard to make use of, so I use this equivalent formulation given in Theorem 3.4.11 as their definitions.

Lemma. *For any bounded sequence (x_n), we always have $\underline{\lim}(x_n) \le \overline{\lim}(x_n)$.*

Theorem (c.f. Theorem 3.4.12 and Section 3.4, Ex.18)**.** *A bounded sequence (x_n) is convergent if and only if $\underline{\lim}(x_n) = \overline{\lim}(x_n)$.*

Proof. ($\Leftarrow$) We can use Squeeze Theorem by noting that $x_n \in \{x_k : k \ge n\}$ for all $n \in \mathbb{N}$, so

$$\inf_{k \ge n} x_k \le x_n \le \sup_{k \ge n} x_k, \quad \forall n \in \mathbb{N}. \tag{1}$$

The result follows at once.

($\Rightarrow$) Let $x = \lim(x_n)$. Taking limit in (1), we know that

$$\inf_{k \ge n} x_k \le \underline{\lim}(x_n) \le x \le \overline{\lim}(x_n) \le \sup_{k \ge n} x_k, \quad \forall n \in \mathbb{N}.$$

Hence x is an upper bound and a lower bound respectively of the sets

$$\left\{\inf_{k\geq n} x_k : n \in \mathbb{N}\right\} \quad \text{and} \quad \left\{\sup_{k\geq n} x_k : n \in \mathbb{N}\right\}. \tag{2}$$

Indeed, we need to show that x is actually the supremum and infimum of these sets respectively. Let $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$|x_k - x| < \frac{\varepsilon}{2}, \quad \forall k \geq n.$$

Equivalently,

$$x - \frac{\varepsilon}{2} < x_k < x + \frac{\varepsilon}{2}, \quad \forall k \geq n.$$

Looking at the two inequalities separately and noting that they hold for all $k \geq n$,

$$x - \varepsilon < x - \frac{\varepsilon}{2} \leq \inf_{k\geq n} x_k \quad \text{and} \quad \sup_{k\geq n} x_k \leq x + \frac{\varepsilon}{2} < x + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that x is the supremum and infimum for the sets in (2) respectively. i.e., $\underline{\lim}(x_n) = x = \overline{\lim}(x_n)$. $\square$

Exercises

Question 1 (c.f. Section 3.2, Ex.15)**.** Show that if $z_n = (a^n + b^n)^{1/n}$ where $0 < a < b$, then

$$\lim(z_n) = b.$$

Solution. We want to use Squeeze Theorem. Since $0 < a < b$,

$$b^n < a^n + b^n < 2b^n, \quad \forall n \in \mathbb{N}.$$

Therefore $b < z_n < b \cdot 2^{1/n}$. Also note that

$$\lim(b) = b \quad \text{and} \quad \lim(b \cdot 2^{1/n}) = b \cdot \lim(2^{1/n}) = b.$$

The result follows. (The calculation above relies on Example 3.1.11(c) and Theorem 3.2.3.)

Question 2 (c.f. Section 3.3, Ex.1)**.** Let $x_1 = 8$ and $x_{n+1} = \frac{1}{2}x_n + 2$ for $n \in \mathbb{N}$. Show that (x_n) is bounded and monotone. Find the limit.

Solution. Let calculate the first few terms to get a feeling about this sequence.

$$(8, 6, 5, 4.5, 4.25, ...)$$

To show that this sequence is bounded, we show $4 \leq x_n \leq 8$ for all $n \in \mathbb{N}$ by induction. The case $n = 1$ is obvious. Now suppose $4 \leq x_n \leq 8$, then

$$\frac{1}{2}(4) + 2 \leq \frac{1}{2}x_n + 2 \leq \frac{1}{2}(8) + 2.$$

i.e., $4 \le x_{n+1} \le 6 \le 8$. Now we show that $x_n - x_{n+1} \ge 0$ for all $n \in \mathbb{N}$, so it is decreasing:

$$x_n - x_{n+1} = x_n - \left(\frac{1}{2}x_n + 2\right) = \frac{1}{2}x_n - 2 \ge \frac{1}{2}(4) - 2 = 0$$

Since this sequence is bounded and monotone, it is convergent by **Monotone Convergence Theorem**. Let $x = \lim(x_n)$ be its limit. Passing the limit into the inductive formula

$$x = \frac{1}{2}x + 2,$$

it follows that the limit is $x = 2$.

Remark. Homework 3: Section 3.3, Q2 is similar.

Question 3 (c.f. Section 3.4, Ex.17)**.** Alternate the terms of the sequences $(1 + 1/n)$ and $(-1/n)$ to obtain the sequence (x_n) given by

$$(2, -1, 3/2, -1/2, 4/3, -1/3, 5/4, -1/4, ...).$$

Determine the values of $\overline{\lim}(x_n)$ and $\underline{\lim}(x_n)$. Also find $\sup x_n$ and $\inf x_n$.

Solution. Let's give the answers before explaining it:

$$\overline{\lim}(x_n) = 1, \quad \underline{\lim}(x_n) = 0 \quad \sup x_n = 2, \quad \text{and} \quad \inf x_n = -1.$$

Note that for each $n \in \mathbb{N}$,

$$\sup_{k \ge n} x_k = 1 + \frac{1}{n_1} \quad \text{and} \quad \inf_{k \ge n} x_k = -\frac{1}{n_2},$$

where n_1 and n_2 are given by

$$n_1 = \left\lfloor \frac{n}{2} \right\rfloor + 1 \quad \text{and} \quad n_2 = \left\lfloor \frac{n+1}{2} \right\rfloor.$$

(Try to visualize the sequence in a graph!) Note that as $n \to \infty$, both n_1 and n_2 also tends to infinity. Hence

$$\overline{\lim}(x_n) = \lim \left(1 + \frac{1}{n}\right) = 1 \quad \text{and} \quad \underline{\lim}(x_n) = \lim \left(-\frac{1}{n}\right) = 0.$$

It is easier to show the supremum and infimum. Observe that

$$1 \le 1 + \frac{1}{n} \le 2 \quad \text{and} \quad -1 \le -\frac{1}{n} \le 0, \quad \forall n \in \mathbb{N}.$$

Hence $\inf x_n \ge -1$ and $\sup x_n \le 2$. As -1 and 2 are terms in (x_n), it follows that $\inf x_n \le -1$ and $\sup x_n \ge 2$.